Screening in the PA model - Continuous Types

Model setup:

Suppose again the principal P is a monopolistic producer of some good or service, and agents A of different types consume this service in variable quantities.

- P's utility is
$$U^P = T - C(x)$$
where x indicates the output level, $C(\cdot)$ the principal's costs for providing the service, and T the agent's monetary payments. We assume $C(0) = 0$, $C_x >$, $C_{xx} \geq 0$, and Inada conditions.
- A's utility:
$$U = u(x, \theta) - T$$
with $u_x > 0$, $u_{xx} < 0$, $u_\theta > 0$, and $u_{x\theta} > 0$ (single crossing property). Reservation utility normalized to zero.
- A's type θ is continuous on interval $[\underline{\theta}, \bar{\theta}]$ according to distribution function $F(\theta)$.
- Monotone hazard rate condition applies:
$$\frac{\partial}{\partial\theta} \frac{1 - F(\theta)}{f(\theta)} \leq 0.$$
Note: This condition on the underlying distribution function is satisfied by most 'prominent' distributions (normal, uniform, chi square etc).

– two additional technical assumptions on 3rd derivatives of $u(\cdot)$ function: $u_{xx\theta} \geq 0$, $u_{\theta\theta x} \leq 0$. (as we will see these conditions are sufficient not necessary).

Analysis:

– The first-best allocation maximizes for each θ the program

$$U^P(\cdot) = T - C(x(\theta)) \quad s.t. \quad U(\cdot) = u(x(\theta), \theta) - T \geq 0$$

Since each agent's (PC) constraint is obviously binding, this program can be rewritten as

$$max_{x(\theta)}\, P = u(x(\theta), \theta) - C(x(\theta)$$

and the FOC implicitly give $x^{FB}(\theta)$ as the solution to

$$V_x(\cdot) = C_x(\cdot, \theta).$$

Result:

1) Output $x^{FB}(\theta)$ monotonically decreasing in θ

2) Transfers $T^{FB}(\theta) = u(x^{FB}(\theta), \theta)$ [note: increasing in θ. WHY?]

– Now: P does not observe θ. We explore the second-best allocation

By the revelation principle: she optimally offers a menu of contracts $\mathbf{Q}(\hat{\theta}) = \{x(\hat{\theta}), T(\hat{\theta})\}$. Output and payments are based on A's announcement of his type, and satisfy

(1) IC: each agent θ prefers 'package' $Q(\theta)$ to any other package $Q(\hat{\theta})$;

(2) PC: each agent θ prefers 'package' $Q(\theta)$ to his default option of rejecting any contract and getting his reservation utility of zero.

BIG ISSUE: Each agent faces a continuum of ICs here! How do we deal with this problem?

==> **Strategy: proceed in 3 steps**

I. Show that IC constraints can be replaced by 'local' incentive constraints. (Key step!)

II. Show that ICs and $PC(\underline{\theta})$ already imply $PC(\theta)$; Show that $PC(\underline{\theta})$ binds.

III. Solve simplified problem (pointwise optimization)

<u>Ad I: Simplifying the IC constraints</u>

Some useful notation first: Let

$$U(\hat{\theta}, \theta) = u(x(\hat{\theta}), \theta) - T(\hat{\theta})$$

be agent θ' s utility if he picks contract designed for agent $\hat{\theta}$

==> can write:

$U(\theta, \theta) \geq U(\hat{\theta}, \theta)$ [IC of agent θ]

$U(\theta, \theta) \geq 0$ [PC of agent θ]

Using this notation, we first show that

$$\frac{dx(\theta)}{d\theta} \geq 0 \tag{1}$$

must hold, i.e., $x(\theta)$ is non-decreasing under a incentive compatible mechanism.

Proof: Suppose by contradiction that for some $\theta' > \theta$, $x(\theta') < x(\theta)$. For this to be true, the set of conditions

$u(x(\theta), \theta) - T(\theta) \geq u(x(\theta'), \theta) - T(\theta')$

$u(x(\theta'), \theta') - T(\theta') \geq u(x(\theta), \theta') - T(\theta)$

must be satisfied. Adding up yields

$u(x(\theta'), \theta') - u(x(\theta), \theta') \geq u(x(\theta'), \theta) - u(x(\theta), \theta)$

which is impossible by SCP. q.e.d.

Next, we derive the 'local' incentive constraints. When choosing from the principal's menu, an agent of type θ solves

$$max_{\hat{\theta}}\ U(\hat{\theta}, \theta) = u(x(\hat{\theta}), \theta) - T(\hat{\theta}).$$

The corresponding FOC is

$$u_x(\cdot)\frac{dx(\hat{\theta})}{d\hat{\theta}} - \frac{dT(\cdot)}{d\hat{\theta}} = 0.$$

Note that by incentive compatibility, this 'local' truthtelling condition must hold at $\hat{\theta} = \theta$, so that

$$u_x(\cdot)\frac{dx(\theta)}{d\theta)} - \frac{dT(\cdot)}{d\theta} = 0 \tag{2}$$

must be satisfied under an incentive compatible mechanism. In what follows we show that that conditions (1) and (2) are not only necessary but also sufficient for truthtelling, i.e., they imply IC.

Proof of sufficiency: Suppose to the contrary that (1) and (2) hold but there exists a $\hat{\theta}$ with the property $U(\hat{\theta}, \theta) > U(\theta, \theta)$. Then, we have

$$U(\hat{\theta}, \theta) - U(\theta, \theta) = \int_{\theta}^{\hat{\theta}} [u_x(x(\tau), \theta)\frac{dx(\tau)}{d\tau} - \frac{dT(\cdot)}{d\tau}]d\tau > 0. \tag{3}$$

Note that $u_x(\cdot)\frac{dx(\theta)}{d\theta} - \frac{dT(\theta)}{d\theta} = 0$ [condition (2)] implies

$$\int_{\theta}^{\hat{\theta}} [u_x(x(\tau), \tau))\frac{dx(\tau)}{d\tau)} - \frac{dT(\cdot)}{d\tau}]d\tau = 0. \tag{4}$$

Suppose that $\theta' > \theta$ (the reverse case is equivalent). Then, SCP implies $u_x(x(\tau), \tau) \geq u_x(x(\tau), \theta)$. In addition, $dx(\tau)/d\tau \geq 0$ [condition 1)] so that the LHS of (4) must exceed the RHS of (3). Accordingly,

$$U(\hat{\theta}, \theta) - U(\theta, \theta) = RHS_{(3)} < LHS_{(4)} = 0,$$

an immediate contradiction. Accordingly, the monotonicity condition (1) plus the local incentive constraints (2) are necessary and sufficient for global incentive compatibility. q.e.d.

(INTUITION?)

STEP II: $PC(\underline{\theta}) = 0$; IC constraints imply $PC(\theta)$ for all θ.

Define $U(\theta) \equiv U(\theta, \theta)$, and note first that

$$U(\theta) = U(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} \frac{dU(\tau, \tau))}{d\tau} d\tau = U(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} u_\tau(x(\tau), \tau) d\tau.$$

We see that $U(\cdot)$ is increasing in type, and hence, the PC constraint of the lowest type must be binding. Accordingly, the optimal contract prescribes $U(\underline{\theta}) = 0$. The integral term can be interpreted as the 'informational rent' which has to be granted to an agent of type θ, for given allocations $x(\theta)$ as proposed by the principal. This means that the choice of outcomes $x(\cdot)$ completely determines informational rents, and payments $T(\theta)$ to each agent under a incentive-compatible mechanism.

STEP III: Optimal contracts.

We can now set up the principal's program. To do so, it is useful to first ignore the monotonicity condition (1), and to analyze the unconstrained maximization problem, which can be written as

$$max_{x(\cdot)}\, U^P = \int_{\underline{\theta}}^{\bar{\theta}} [u(x(\theta), \theta) - C(x(\theta)) - [\int_{\underline{\theta}}^{\theta} u_\tau(x(\tau), \tau) d\tau]\, f(\theta)\, d\theta. \quad (5)$$

The Principal chooses outputs $x(\theta)$ in a way to maximize her expected utility: this is expected total surplus, minus the expected information rents that have to be granted to the agent under an incentive-compatible contract.

At first sight, this program looks quite nasty, but fortunately we can bring it into a more tractable form. Specifically, consider the expected informational rent

$$\int_{\underline{\theta}}^{\bar{\theta}}[\int_{\underline{\theta}}^{\theta} u_\tau(x(\tau),\tau)d\tau]f(\theta)d\theta.$$

To simplify this expression, we can now use integration by parts i.e., apply the formula

$$\int_a^b z(x)v'(x) = z(x)v(x)|_b^a - \int_a^b z'(x)v(x)$$

Let $z \equiv \int_{\underline{\theta}}^{\theta} u_\tau(x(\tau),\tau)d\tau$ and $v' \equiv f(\theta)$. Noting that

$$z(x)v(x) = \int_{\underline{\theta}}^{\bar{\theta}} u_\tau(x(\tau),\tau)d\tau]F(\bar{\theta}) - [...]F(\underline{\theta}) = \int_{\underline{\theta}}^{\bar{\theta}} u_\theta(x(\theta),\theta)d\theta$$

and

$$\int_a^b z'(x)v(x) = \int_{\underline{\theta}}^{\bar{\theta}} u_\theta(\cdot)F(\theta)d(\theta),$$

one obtains

$$\int_{\underline{\theta}}^{\bar{\theta}}[\int_{\underline{\theta}}^{\theta} u_\tau(x(\tau),\tau)d\tau]f(\theta)d\theta = \int_{\underline{\theta}}^{\bar{\theta}} u_\theta(\cdot)\frac{1-F(\theta)}{f(\theta)}f(\theta)d\theta.$$

Using this last expression in the principal's optimization problem, the program can now be written in its final form as

$$max_{x(\cdot)}\ U^P = \int_{\underline{\theta}}^{\bar{\theta}}[u(x(\theta),\theta) - C(x(\theta)) - u_\theta(\cdot)\frac{1-F(\theta)}{f(\theta)}f(\theta)]d\theta. \quad (6)$$

Pointwise differentiation of U^P with respect to $x(\theta)$ yields

$$u_x(x(\theta),\theta) = C_x(x(\theta)) + u_{\theta x}\frac{1-F(\theta)}{f(\theta)}. \quad (7)$$

Results and Interpretation:

1) The allocative outcome is downwards distorted for all but the most efficient type ('no distortion at the top'- property). Technically: under the single crossing property $u_{x\theta} > 0$, the RHS of (7) is larger than C_x unless $\theta = \bar{\theta}$ where $1 - F(\theta) = 0$. Intuitively: solution $x^*(\theta)$ must be smaller than $x^{FB}(\theta)$ because the 'virtual' marginal costs include informational rents in the second-best problem.

2) The first-order condition (7) indeed represents the solution to the principal's program if the associated second order conditions hold, and in addition the monotonicity condition $dx(\theta)/d(\theta) \geq 0$ is satisfied. One can check: these conditions hold under the monotone hazard rate condition, and if the technical conditions $\partial^3 u(x(\theta), \theta)/\partial x \partial \theta^2 \leq 0$ and $\partial^3 u(x(\theta), \theta)/\partial \theta \partial x^2 \geq 0$ are satisfied (omitted here).

3) The agent's informational rent $U(\theta)$ is increasing in his type. His monetary payment is $T(\theta) = -u(x^*(\theta), \theta) - U(\theta)$.

4) Fundamental tradeoff between rent extraction and allocative efficiency. Raising the output of some lower type boosts the information rent that has to be granted to ALL higher types (reason: it becomes more attractive for higher types to mimic this guy).

Larger $f(\theta)$: more agents of type θ ==> distortion c.p. smaller as allocative efficiency becomes more important

Smaller $F(\theta)$: more agents 'above' agent θ => rent extraction becomes more important, larger distortion of $x(\cdot)$.

6) Possible Problems:

– If Monotone hazard rate condition not satisfied, some types may have to be pooled(obtain the same contract). Technically complicated. Same if reservation utilities are type dependent.

– Renegotiation: after agent truthfully θ, both P and A have incentive to renegotiate to a Pareto-efficient outcome. But then: incentive compatibility is lost. Possible way out (in some situations): replace direct revelation mechanism by mechanism where agent reveals his type only ex post, through the consumption decision.